The Generalized Symmetry Method for Discrete Equations

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Abstract

The generalized symmetry method is applied to a class of completely discrete equations including the Adler-Bobenko-Suris list. Assuming the existence of a generalized symmetry, we derive a few integrability conditions suitable for testing and classifying equations of this class. Those conditions are used at the end to test for integrability discretizations of some well-known hyperbolic equations.

1 Introduction

The discovery of new two-dimensional integrable partial difference equations (or $\mathbb{Z}^2$-lattice equations) is always a very challenging problem as, by proper continuous limits, many other results on integrable differential-difference and partial differential equations may be obtained.

The basic theory and results in nonlinear integrable differential equations can be found, for example, in the *Encyclopedia of Mathematical Physics* [16] or in the *Encyclopedia of Nonlinear Science* [17].

The classification of integrable nonlinear partial differential equations has been widely discussed in many relevant papers. Let us just mention here the classification

scheme introduced by Shabat, where the formal symmetry approach has been introduced (see [30,31] for a review). This approach has been successfully extended to the differential-difference case by Yamilov [7, 28, 45, 46]. In the completely discrete case the situation turns out to be quite different, and, up to now, the formal symmetry technique has not been able to provide any result. In the case of difference–difference equations, the first exhaustive classification has been obtained in [4] by Adler and in [5,6,9,12] by Adler, Bobenko and Suris for linear affine equations. The so obtained equations have been thoroughly studied by many researchers, and it has been shown that they have Lax pairs and that possess generalized symmetries [26, 33, 35, 40, 41].

We study in this paper the following class of autonomous discrete equations on the lattice $\mathbb{Z}^2$:

$$u_{i+1,j+1} = F(u_{i+1,j}, u_{i,j}, u_{i,j+1}), \tag{1}$$

where i, j are arbitrary integers. Many integrable examples of equations of this form are known [5, 24, 25, 41, 42]. There are a number of papers, in which various schemes for testing and classifying integrable equations of the form (1) are discussed [2, 5, 8, 11, 18–20, 34, 37]. In [8, 18] the classification of linearizable equations is considered, in [2,11,19,20,34] extensions of the Painlevé test are carried over to the discrete case while in [5, 37] equations integrable by the inverse scattering method are discussed. Requiring additional geometrical symmetry properties, a classification result has been obtained in [5] together with a list of integrable equations. However, the symmetries for those discrete equations obtained in [35, 41] show that the obtained class of equations contained in [5] is somehow restricted [26]. From [26] it follows that one should expect a larger number of integrable discrete equations of the kind of eq. (1) than those up to now known.

Eqs. (1) are possible discrete analogs of the hyperbolic equations

$$u_{x,y} = F(u_x, u, u_y). \tag{2}$$

Eqs. (2) are very important in many fields of physics, and, as such, they have been studied using the generalized symmetry method, however without much success. Only the following two particular cases:

$$u_{x,y} = F(u), \tag{3}$$

$$u_x = F(u, v), \quad v_y = G(u, v), \tag{4}$$

which are essentially easier, have been solved [47, 48]. The study of the class of equations (1) may be important to characterize the integrable subcases of eq. (2).

In Section 2 we introduce and discuss some necessary notions, such as generalized symmetries and conservation laws for discrete systems of the form (1), and in Section 3, as a motivation for the use of this approach, we show that one can construct a partial difference equation closely related to the modified Volterra equation, which does not belong to the ABS class of equations as it is not 3D–consistent around the cube and does not have the D_4 symmetry. In Section 4, following the standard

scheme of the generalized symmetry method, we derive a few integrability conditions for the class (1). These conditions are not sufficient to carry out a classification of the discrete equations (1). So in Section 5 we reduce ourselves to consider just 5 points generalized symmetries. This request provides further integrability conditions. With these extra conditions, the set of obtained conditions will be suitable for testing and classifying simple classes of difference equations of the form (1). As an example, in Section 6 we apply those conditions to the class of equations

$$u_{i+1,j+1} = u_{i+1,j} + u_{i,j+1} + \varphi(u_{i,j}), \tag{5}$$

a trivial approximation to the class (3). This calculation is an example of classification problem for a class depending on one unknown function of one variable. This class contains trivial approximations of some well-known integrable equations included in the class (3), namely, the sine-Gordon, Tzitzèika and Liouville equations. Section 7 contains some conclusive remarks.

2 Preliminary definitions

As eq. (1) has no explicit dependence on the point (i, j) of the lattice, we assume that the same will be for the generalized symmetries and conservation laws we will be considering in the following. For this reason, without loss of generality, we write down symmetries and conservation laws at the point $(0, 0)$. Thus eq. (1) can be written as:

$$u_{1,1} = f_{0,0} = F(u_{1,0}, u_{0,0}, u_{0,1}). \tag{6}$$

Whenever convenient we will express our formulas in terms of the two shift operators, T_1 and T_2:

$$T_1 u_{i,j} = u_{i+1,j}, \qquad T_2 u_{i,j} = u_{i,j+1}. \tag{7}$$

To get a scheme which is invertible and to provide propagation in both discrete directions, we have to suppose that the function F depends on all its variables, i.e.

$$\partial_{u_{1,0}} F \ \cdot \partial_{u_{0,0}} F \ \cdot \partial_{u_{0,1}} F \neq 0. \tag{8}$$

The functions $u_{i,j}$ are related among themselves by eq. (6) and its shifted values

$$u_{i+1,j+1} = T_1^i T_2^j f_{0,0} = f_{i,j} = F(u_{i+1,j}, u_{i,j}, u_{i,j+1}),$$

and it is easy to see that all of them can be expressed in terms of the functions

$$u_{i,0}, \quad u_{0,j}, \tag{9}$$

where i, j are arbitrary integers. This is not the only possible choice of independent variables [10], but, being the simplest, is the one we will use in the following. The functions (9) play the role of boundary–initial conditions for eq. (6).

The evolutionary form of a generalized symmetry of eq. (6) is given by the following equation

$$\frac{d}{dt}u_{0,0} = g_{0,0} = G(u_{n,0}, u_{n-1,0}, \ldots, u_{n',0}, u_{0,k}, u_{0,k-1}, \ldots, u_{0,k'}), \tag{10}$$

where $n \geq n'$, $k \geq k'$. The form of this equation at the various points of the lattice is obtained by applying the shift operators T_1 and T_2:

$$\frac{d}{dt}u_{i,j} = T_1^i T_2^j g_{0,0} = g_{i,j} = G(u_{i+n,j}, \ldots, u_{i+n',j}, u_{i,j+k}, \ldots, u_{i,j+k'}).$$

Eq. (10) is a generalized symmetry of eq. (6) if the two equations (6, 10) are compatible for all independent variables (9), i.e.

$$\frac{du_{1,1}}{dt} - \frac{df_{0,0}}{dt}\bigg|_{u_{1,1}=f_{0,0}} = 0. \tag{11}$$

In practice, eq. (11) reads

$$g_{1,1} = (g_{1,0}\partial_{u_{1,0}} + g_{0,0}\partial_{u_{0,0}} + g_{0,1}\partial_{u_{0,1}})f_{0,0}. \tag{12}$$

Eq. (12) must be identically satisfied when all the variables $u_{i,j}$ contained in the functions $g_{i,j}$ and in the derivatives of $f_{0,0}$ are expressed in terms of the independent variables (9). This result provides strict conditions, given by a set of equations for the functions F and G, often overdetermined.

Let us consider some autonomous functions $p_{0,0}, q_{0,0}$ which depend on a finite number of functions $u_{i,j}$ and have no explicit dependence on the point (i,j) of the lattice. The relation

$$(T_1 - 1)p_{0,0} = (T_2 - 1)q_{0,0} \tag{13}$$

is called a (local i,j-independent) conservation law of eq. (6) if it is satisfied on the solutions set of this equation. To check it, we need to express all variables in terms of the independent variables (9) and require that it is identically satisfied.

Starting from the choice of the independent variables (9) and the class of autonomous difference equations (6), we can prove a few useful statements which will be used for studying the compatibility condition (12). Let us consider the functions $u_{i,1}, u_{1,j}$ appearing in eq. (12). We can prove the following Theorem:

Theorem 1 *The transformation* $\mathcal{T} : \{u_{i,0}, u_{0,j}\} \to \{\tilde{u}_{i,0}, \tilde{u}_{0,j}\}$*, given by the shift operator* T_2

$$\tilde{u}_{0,j} = u_{0,j+1}, \qquad \tilde{u}_{i,0} = u_{i,1}, \quad i \neq 0, \tag{14}$$

is invertible under the equation (6). Moreover, if a function ϕ *is non–zero, then* $T_2\phi \neq 0$ *too.*

Proof. The invertibility of the transformation $\tilde{u}_{0,j} = u_{0,j+1}$ is obvious. Let us show by induction that for any $i \geq 1$

$$\tilde{u}_{i,0} = \tilde{u}_{i,0}(u_{i,0}, u_{i-1,0}, \ldots, u_{1,0}, u_{0,0}, u_{0,1}), \quad \partial_{u_{i,0}}\tilde{u}_{i,0} \neq 0, \quad \partial_{u_{0,1}}\tilde{u}_{i,0} \neq 0. \tag{15}$$

It follows from eq. (6) and condition (8) that the proposition is true for $\tilde{u}_{1,0} = u_{1,1}$. For $i \geq 1$, from eq. (6) we get

$$\tilde{u}_{i+1,0} = u_{i+1,1} = F(u_{i+1,0}, u_{i,0}, \tilde{u}_{i,0}), \tag{16}$$

with $\tilde{u}_{i,0}$ given by eq. (15). So $\tilde{u}_{i+1,0}$ has the same structure as $\tilde{u}_{i,0}$ and thus eq. (15) is true. As $\tilde{u}_{i,0}$ depends on $u_{0,1}$, then the functions $u_{i+1,0}, u_{i,0}, \tilde{u}_{i,0}$ are functionally independent, i.e. $\partial_{u_{i+1,0}}\tilde{u}_{i+1,0} \neq 0$ and $\partial_{u_{0,1}}\tilde{u}_{i+1,0} \neq 0$. A similar analysis can be carried out in the case of the functions $\tilde{u}_{i,0}$ when $i \leq -1$. In this case we have

$$\tilde{u}_{i,0} = \tilde{u}_{i,0}(u_{i,0}, u_{i+1,0}, \ldots, u_{-1,0}, u_{0,0}, u_{0,1}), \qquad \partial_{u_{i,0}}\tilde{u}_{i,0} \neq 0, \quad \partial_{u_{0,1}}\tilde{u}_{i,0} \neq 0. \tag{17}$$

From eqs. (15, 17) it follows that the transformation (14) is invertible.

To prove the second part of this Theorem, let us consider a non-constant function $\phi \neq 0$. Taking into account eq. (6) and its shifted values, ϕ can always be expressed in terms of the independent variables as

$$\phi = \Phi(u_{N,0}, u_{N-1,0}, \ldots, u_{N',0}, u_{0,K}, u_{0,K-1}, \ldots, u_{0,K'}), \tag{18}$$

for some integer numbers N, N', K and K' such that $N \geq N'$, $K \geq K'$. Then we will have

$$T_2\phi = \Phi(\tilde{u}_{N,0}, \ldots, \tilde{u}_{N',0}, \tilde{u}_{0,K}, \ldots, \tilde{u}_{0,K'}). \tag{19}$$

If ϕ depends essentially on the variables $u_{i,0}$ with $i \neq 0$, then there must exist two numbers N and N' such that $\partial_{u_{N,0}}\phi \neq 0$ and $\partial_{u_{N',0}}\phi \neq 0$. When $N > 0$, from eq. (15) it follows that only the function $\tilde{u}_{N,0}$ appearing in eq. (19) depends on $u_{N,0}$. Hence $\partial_{u_{N,0}}T_2\phi \neq 0$, i.e. $T_2\phi \neq 0$. The case, when $N' < 0$, is analogous. If ϕ depends only on $u_{0,j}$, then $\partial_{u_{0,K}}\phi \neq 0$ and $\partial_{u_{0,K'}}\phi \neq 0$, and the proof is obvious. ∎

The operators T_1, T_1^{-1}, T_2^{-1} act on the variables (9) in an analogous way. Consequently they also define invertible transformations. As a result we can state the following Proposition:

Proposition 1 *For any non-zero function ϕ, $T_1^l T_2^m \phi \neq 0$ for any $l, m \in \mathbb{Z}$.*

From eqs. (14, 15, 17) we can derive the structure of some of the partial derivatives of the functions $u_{i,1}$. For convenience, from now on we will define

$$f_{u_{i,j}} = \partial_{u_{i,j}} f_{0,0}, \qquad g_{u_{i,j}} = \partial_{u_{i,j}} g_{0,0} \tag{20}$$

for the derivatives of the functions $f_{0,0}$ and $g_{0,0}$ appearing in eqs. (6, 10). Then, for example, from eq. (6) we get $\partial_{u_{1,0}}u_{1,1} = f_{u_{1,0}}$. For $i > 0$, from eqs. (15, 16)

it follows that $\partial_{u_{i+1,0}} u_{i+1,1} = T_1^i \partial_{u_{1,0}} u_{1,1}$. From eq. (6) we can also get $u_{-1,1} = \hat{F}(u_{-1,0}, u_{0,0}, u_{0,1})$ and then by differentiation

$$\partial_{u_{-1,0}} u_{-1,1} = -T_1^{-1} \frac{f_{u_{0,0}}}{f_{u_{0,1}}}. \tag{21}$$

Then, applying the operator T_1^{i+1}, with $i < 0$, to eq. (21) it follows that $\partial_{u_{i,0}} u_{i,1} = -T_1^i \frac{f_{u_{0,0}}}{f_{u_{0,1}}}$. For the functions of the form $u_{1,j}$ we get similar results. So we can state the following Proposition:

Proposition 2 *The functions $u_{i,1}, u_{1,j}$ are such that*

$$\begin{array}{lll} i > 0: & u_{i,1} = u_{i,1}(u_{i,0}, u_{i-1,0}, \dots, u_{1,0}, u_{0,0}, u_{0,1}), & \partial_{u_{i,0}} u_{i,1} = T_1^{i-1} f_{u_{1,0}}; \\ i < 0: & u_{i,1} = u_{i,1}(u_{i,0}, u_{i+1,0}, \dots, u_{-1,0}, u_{0,0}, u_{0,1}), & \partial_{u_{i,0}} u_{i,1} = -T_1^i \frac{f_{u_{0,0}}}{f_{u_{0,1}}}; \\ j > 0: & u_{1,j} = u_{1,j}(u_{1,0}, u_{0,0}, u_{0,1}, \dots, u_{0,j-1}, u_{0,j}), & \partial_{u_{0,j}} u_{1,j} = T_2^{j-1} f_{u_{0,1}}; \\ j < 0: & u_{1,j} = u_{1,j}(u_{1,0}, u_{0,0}, u_{0,-1}, \dots, u_{0,j+1}, u_{0,j}), & \partial_{u_{0,j}} u_{1,j} = -T_2^j \frac{f_{u_{0,0}}}{f_{u_{1,0}}}. \end{array} \tag{22}$$

3 Integrable example

In this Section we show, using a simple example, that effectively there are integrable equations which possess hierarchies of generalized symmetries of the form postulated in eq. (10) and are not included in the ABS lists.

As it is well-known [43], the modified Volterra equation

$$u_{i,t} = (u_i^2 - 1)(u_{i+1} - u_{i-1}) \tag{23}$$

is transformed into the Volterra equation $v_{i,t} = v_i(v_{i+1} - v_{i-1})$ by two discrete Miura transformations:

$$v_i^{\pm} = (u_{i+1} \pm 1)(u_i \mp 1). \tag{24}$$

For any solution u_i of eq. (23), one obtains by the transformations (24) two solutions v_i^+, v_i^- of the Volterra equation. From a solution of the Volterra equation v_i one obtains two solutions u_i and $\tilde{u}_i$ of the modified Volterra equation. The composition of the Miura transformations (24)

$$v_i = (u_{i+1} + 1)(u_i - 1) = (\tilde{u}_{i+1} - 1)(\tilde{u}_i + 1) \tag{25}$$

provides a Bäcklund transformation for eq. (23). Eq. (25) allows one to construct, starting with a solution u_i of the modified Volterra equation (23), a new solution $\tilde{u}_i$.

Introducing for any index i $u_i = u_{i,j}$ and $\tilde{u}_i = u_{i,j+1}$, where j is a new index, we can rewrite the Bäcklund transformation (25) as an equation of the form (1). At the point $(0,0)$ it reads:

$$(u_{1,0} + 1)(u_{0,0} - 1) = (u_{1,1} - 1)(u_{0,1} + 1). \tag{26}$$

Eq. (26) does not belong to the ABS classification, as it is not invariant under the exchange of i and j and does not satisfy the 3D–consistency property [29]. The modified Volterra equation (23) can then be interpreted as a 3 points generalized symmetry of eq. (26) involving only shifts in the i direction:

$$u_{0,0,t} = (u_{0,0}^2 - 1)(u_{1,0} - u_{-1,0}). \tag{27}$$

There exists also a generalized symmetry involving only shifts in the j direction, given by

$$u_{0,0,\tau} = (u_{0,0}^2 - 1)\left(\frac{1}{u_{0,1} + u_{0,0}} - \frac{1}{u_{0,0} + u_{0,-1}}\right), \tag{28}$$

which belongs, together with eq. (27), to the complete list of integrable Volterra type equations presented in [45, 46]. Both equations have a hierarchy of generalized symmetries which, by construction, must be compatible with eq. (26). Symmetries of eq. (27) can be obtained in many ways, see e.g. [46]. Symmetries of eq. (28) can be constructed, using the master symmetry presented in [14]. The simplest generalized symmetries of eqs. (27) and (28) are given by the following equations:

$$\begin{aligned}
u_{0,0,t'} &= (u_{0,0}^2 - 1)((u_{1,0}^2 - 1)(u_{2,0} + u_{0,0}) - (u_{-1,0}^2 - 1)(u_{0,0} + u_{-2,0})),\\
u_{0,0,\tau'} &= \frac{u_{0,0}^2 - 1}{(u_{0,1} + u_{0,0})^2}\left(\frac{u_{0,1}^2 - 1}{u_{0,2} + u_{0,1}} + \frac{u_{0,0}^2 - 1}{u_{0,0} + u_{0,-1}}\right)\\
&- \frac{u_{0,0}^2 - 1}{(u_{0,0} + u_{0,-1})^2}\left(\frac{u_{0,0}^2 - 1}{u_{0,1} + u_{0,0}} + \frac{u_{0,-1}^2 - 1}{u_{0,-1} + u_{0,-2}}\right).
\end{aligned}$$

As it can be checked by direct calculation, these equations are 5 points symmetries of eq. (26).

Moreover, eq. (26) possesses two conservation laws (13) characterized by the following functions $p_{0,0}, q_{0,0}$:

$$p_{0,0}^+ = \log \tfrac{u_{0,0}+u_{0,1}}{u_{0,0}+1}, \qquad q_{0,0}^+ = -\log(u_{0,0} + 1), \tag{29}$$

$$p_{0,0}^- = \log \tfrac{u_{0,0}+u_{0,1}}{u_{0,1}-1}, \qquad q_{0,0}^- = \log(u_{0,0} - 1). \tag{30}$$

It is easy to check that eq. (13) is identically satisfied on the solutions of eq. (26) when we introduce into it the functions (29) or (30). Eq. (26) possess also non–autonomous conservation laws, however, conservation laws of this kind will not be discussed here.

A more general form of both eqs. (25, 26) is given by

$$v_{i,j} = (u_{i+1,j} + \alpha_j)(u_{i,j} - \alpha_j) = (u_{i+1,j+1} - \alpha_{j+1})(u_{i,j+1} + \alpha_{j+1}), \tag{31}$$

where α_j is a j-dependent function. For any j the function $u_{i,j}$ satisfies the modified Volterra equation

$$u_{i,j,t} = (u_{i,j}^2 - \alpha_j^2)(u_{i+1,j} - u_{i-1,j})$$

depending on the function α_j. Function $v_{i,j}$, for any j, is a solution of the Volterra equation. Using eq. (31) and starting from an initial solution $v_{i,0}$, we can construct new solutions of the Volterra equation:

$$v_{i,0} \to u_{i,1} \to v_{i,1} \to u_{i,2} \to v_{i,2} \to \dots \ .$$

The Lax pair for eq. (31) is given by

$$L_{i,j} = \begin{pmatrix} \lambda - \lambda^{-1} & -v_{i,j} \\ 1 & 0 \end{pmatrix},$$

which corresponds to the standard scalar spectral problem of the Volterra equation written in matrix form, and by

$$A_{i,j} = \frac{1}{u_{i,j+1} - \alpha_{j+1}} \begin{pmatrix} (\lambda - \lambda^{-1})(u_{i,j+1} - \alpha_{j+1}) & 2\alpha_{j+1}(u_{i,j+1}^2 - \alpha_{j+1}^2) \\ -2\alpha_{j+1} & (\lambda - \lambda^{-1})(u_{i,j+1} + \alpha_{j+1}) \end{pmatrix}.$$

This Lax pair satisfies the Lax equation $A_{i+1,j}L_{i,j} = L_{i,j+1}A_{i,j}$. By setting $\alpha_j = 1$ we get a Lax pair for eq. (26). A different Lax pair for this equation has been constructed in [29].

Eq. (31) is a direct analog of well-known dressing chain

$$u_{j+1,x} + u_{j,x} = u_{j+1}^2 - u_j^2 + \alpha_{j+1} - \alpha_j \tag{32}$$

which provides a way of constructing potentials $v_j = u_{j,x} - u_j^2 - \alpha_j$ for the Schrödinger spectral problem [38, 39]. The Lax pair given above is analogous to that of eq. (32) presented in [39].

4 Derivation of the integrability conditions

In this Section, following the standard scheme of the generalized symmetry method, we derive from the compatibility condition (12) four conditions necessary for the integrability of eq. (6).

For a generalized symmetry (10) we suppose that if $g_{0,0}$ depends on at least one variable of the form $u_{i,0}$, then $g_{u_{n,0}} \neq 0$ and $g_{u_{n',0}} \neq 0$, and the numbers n, n' are called orders of the symmetry. The same can be said about the variables $u_{0,j}$ and the corresponding numbers k, k' if $g_{u_{0,k}} \neq 0$ and $g_{u_{0,k'}} \neq 0$.

Theorem 2 *Let eq. (6) possess a generalized symmetry (10) of orders n, n', k and k'. Then the following relations must take place:*

$$\text{If} \quad n > 0 \quad \Longrightarrow \quad (T_1^n - 1) \log f_{u_{1,0}} = (1 - T_2) T_1 \log g_{u_{n,0}}; \tag{33}$$

$$\text{If} \quad n' < 0 \quad \Longrightarrow \quad (T_1^{n'} - 1) \log \frac{f_{u_{0,0}}}{f_{u_{0,1}}} = (1 - T_2) \log g_{u_{n',0}}; \tag{34}$$

$$\text{If} \quad k > 0 \quad \Longrightarrow \quad (T_2^k - 1) \log f_{u_{0,1}} = (1 - T_1) T_2 \log g_{u_{0,k}}; \tag{35}$$

$$\text{If} \quad k' < 0 \quad \Longrightarrow \quad (T_2^{k'} - 1) \log \frac{f_{u_{0,0}}}{f_{u_{1,0}}} = (1 - T_1) \log g_{u_{0,k'}}. \tag{36}$$

Before going over to the proof of this Theorem, let us clarify its meaning by noting that in the case of a three point symmetry with $g_{0,0} = G(u_{1,0}, u_{0,0}, u_{-1,0})$, for which $n > 0$ and $n' < 0$, one can use both relations (33, 34).

Proof. Let us consider the compatibility condition (12) expressed in terms of the independent variables (9). As $g_{0,0}$ depends on $u_{i,0}$ and $u_{0,j}$, the functions $(g_{1,1}, g_{1,0}, g_{0,1})$ depend on $(u_{i,1}, u_{1,j})$, whose form is given by Proposition 2. Moreover, eq. (12) will contain $u_{i,0}$ with $n+1 \geq i \geq n'$ and $u_{0,j}$ with $k+1 \geq j \geq k'$.

If $n > 0$, applying to eq. (12) the operator $\partial_{u_{n+1,0}}$ and using the results (22) contained in Proposition 2, we get:

$$T_1 T_2 (g_{u_{n,0}}) T_1^n f_{u_{1,0}} = f_{u_{1,0}} T_1 g_{u_{n,0}}.$$

Applying the logarithm to both sides of the previous equation, we obtain eq. (33). The other cases are obtained in a similar way by differentiating eq. (12) with respect to $u_{n',0}$, $u_{0,k+1}$, and $u_{0,k'}$. ∎

Eqs. (33–36) can be expressed as a standard conservation law of the form (13), using the obvious well-known formulae:

$$\begin{aligned} T_l^m - 1 &= (T_l - 1)(1 + T_l + \ldots + T_l^{m-1}), \qquad m > 0,\\ T_l^m - 1 &= (1 - T_l)(T_l^{-1} + T_l^{-2} + \ldots + T_l^m), \quad m < 0, \end{aligned}$$

where $l = 1, 2$. This means that, from the existence of a generalized symmetry, one can construct some conservation laws.

Theorem 2 provides integrability conditions, i.e. that for an integrable equation there must exist a function $g_{0,0}$ satisfying eqs. (33–36). The unknown function $g_{0,0}$ must depend on a finite number of independent variables. These integrability conditions turn out to be difficult to use for testing and classifying difference equations.

In the case of the differential-difference equations of Volterra or Toda type [46], there are integrability conditions equivalent to eqs. (33–36). In order to check these integrability conditions one can use the formal variational derivatives [15,22,44,46], defined as

$$\frac{\delta^{(1)}\phi}{\delta u_{0,0}} = \sum_{i=-N}^{-N'} \frac{\partial T_1^i \phi}{\partial u_{0,0}}, \qquad \frac{\delta^{(2)}\phi}{\delta u_{0,0}} = \sum_{j=-K}^{-K'} \frac{\partial T_2^j \phi}{\partial u_{0,0}},$$

for ϕ given by eq. (18). Using such variational derivatives, for example the integrability conditions (33, 35) are reduced to the following equations:

$$\frac{\delta^{(2)}}{\delta u_{0,0}}(T_1^n - 1)\log f_{u_{1,0}} = 0, \qquad \frac{\delta^{(1)}}{\delta u_{0,0}}(T_2^k - 1)\log f_{u_{0,1}} = 0, \tag{37}$$

which do not involve any unknown function. This result is due to the fact that in this case all discrete variables are independent. In the completely discrete case

the situation is essentially different. Some of the discrete variables are dependent and the variational derivatives must be calculated modulo the equation (1). So, eqs. (37) will not be anymore valid. If we apply here the variational derivatives, we will get, at most, some partial results depending on the choice of the independent variables introduced.

The conservation laws (33–36) depend on the order of the symmetry. These conservation laws can be simplified under some assuptions on the structure of the Lie algebra of the generalized symmetries. If we assume that for a given equation we are able to get generalized symmetries for any value of n and k, then we can derive order-independent conservation laws, using a trick standard in the generalized symmetry method [46]. This assumption implies that if, for example, we have a generalized symmetry of order n then there must be also one of order $n+1$. This is a very constraining assumption which is not always verified, as we know from the continuous case [31]. Here it is used just as an example for the construction of simplified formulas. In fact such simplified formulas can be obtained assuming any difference between the orders of two generalized symmetries, and in next Section we consider an example with difference 2.

So, in the following Theorem, we will assume that in addition to (10) a second generalized symmetry

$$u_{0,0,\tilde{t}} = \tilde{g}_{0,0} = \tilde{G}(u_{\tilde{n},0}, u_{\tilde{n}-1,0}, \ldots, u_{\tilde{n}',0}, u_{0,\tilde{k}}, u_{0,\tilde{k}-1}, \ldots, u_{0,\tilde{k}'}) \tag{38}$$

of orders $\tilde{n}, \tilde{n}', \tilde{k}, \tilde{k}'$ will exist. With this assumption we shall obtain four conservation laws:

$$(T_1 - 1)p_{0,0}^{(m)} = (T_2 - 1)q_{0,0}^{(m)}, \qquad m = 1, 2, 3, 4, \tag{39}$$

with $p_{0,0}^{(m)}$ or $q_{0,0}^{(m)}$ expressed in terms of eq. (6).

Theorem 3 *Let eq. (6) possess two generalized symmetries (38) and (10). Then eq. (6) admits the conservation laws (39):*

$$n > 0,\ \tilde{n} = n+1 \qquad \Longrightarrow \qquad m = 1, \quad p_{0,0}^{(1)} = \log f_{u_{1,0}}; \tag{40}$$

$$n' < 0,\ \tilde{n}' = n'-1 \qquad \Longrightarrow \qquad m = 2, \quad p_{0,0}^{(2)} = \log \frac{f_{u_{0,0}}}{f_{u_{0,1}}}; \tag{41}$$

$$k > 0,\ \tilde{k} = k+1 \qquad \Longrightarrow \qquad m = 3, \quad q_{0,0}^{(3)} = \log f_{u_{0,1}}; \tag{42}$$

$$k' < 0,\ \tilde{k}' = k'-1 \qquad \Longrightarrow \qquad m = 4, \quad q_{0,0}^{(4)} = \log \frac{f_{u_{0,0}}}{f_{u_{1,0}}}. \tag{43}$$

Proof. Let us consider in detail just the case when $n > 0,\ \tilde{n} = n+1$. Due to Theorem 2 eq. (33) must be satisfied and consequently

$$(T_1^{n+1} - 1)p_{0,0}^{(1)} = (1 - T_2)T_1 \log \tilde{g}_{u_{n+1,0}}, \tag{44}$$

where $p_{0,0}^{(1)}$ is given by (40). Applying the operator $-T_1$ to eq. (33) and adding the result to eq. (44), we get the conservation law (39) with $m=1$, where $q_{0,0}^{(1)}$ is given by:

$$q_{0,0}^{(1)} = T_1^2 \log g_{u_{n,0}} - T_1 \log \tilde{g}_{u_{n+1,0}}.$$

The other cases are proved in an analogous way. ∎

So for eq. (6) we have four necessary conditions of integrability: there must exist some functions of finite range $q_{0,0}^{(1)}, q_{0,0}^{(2)}, p_{0,0}^{(3)}, p_{0,0}^{(4)}$ of the form (18) satisfying the conservation laws (39) with $p_{0,0}^{(1)}, p_{0,0}^{(2)}, q_{0,0}^{(3)}, q_{0,0}^{(4)}$ defined by eq. (40–43).

The following Theorem will precise the structure of the unknown functions $q_{0,0}^{(1)}$, $q_{0,0}^{(2)}$, $p_{0,0}^{(3)}$, and $p_{0,0}^{(4)}$.

Theorem 4 *If the functions $q_{0,0}^{(1)}$, $q_{0,0}^{(2)}$, $p_{0,0}^{(3)}$, and $p_{0,0}^{(4)}$ satisfy eq. (39), with $p_{0,0}^{(1)}$, $p_{0,0}^{(2)}$, $q_{0,0}^{(3)}$, and $q_{0,0}^{(4)}$ given by eqs. (40–43), and are written in the form (18), then $q_{0,0}^{(1)}$ and $q_{0,0}^{(2)}$ may depend only on the variables $u_{i,0}$, and $p_{0,0}^{(3)}$ and $p_{0,0}^{(4)}$ on $u_{0,j}$.*

Proof. Let us consider eq. (39) with $m=1$. The functions therein involved have the following form:

$$\begin{aligned}
p_{0,0}^{(1)} &= P^{(1)}(u_{1,0}, u_{0,0}, u_{0,1}), \qquad p_{1,0}^{(1)} = P^{(1)}(u_{2,0}, u_{1,0}, u_{1,1}),\\
q_{0,0}^{(1)} &= Q^{(1)}(u_{N,0}, \ldots, u_{N',0}, u_{0,K}, \ldots, u_{0,K'}),\\
q_{0,1}^{(1)} &= Q^{(1)}(u_{N,1}, \ldots, u_{N',1}, u_{0,K+1}, \ldots, u_{0,K'+1}).
\end{aligned}$$

Let us consider the function $q_{0,0}^{(1)}$ and let us study its dependence on the variables $u_{0,j}$ with $j \neq 0$. Using Proposition 2, we see that the functions $u_{i,1}$ in $p_{1,0}^{(1)}, q_{0,1}^{(1)}$ may depend only on $u_{0,1}$. If $K > 0$, we differentiate eq. (39) with $m=1$ with respect to $u_{0,K+1}$ and get: $\partial_{u_{0,K+1}} q_{0,1}^{(1)} = T_2 \partial_{u_{0,K}} q_{0,0}^{(1)} = 0$. Then, from Proposition 1, it follows that $q_{0,0}^{(1)}$ does not depend on $u_{0,K}$. If $K' < 0$, let us differentiate with respect to $u_{0,K'}$ and we get $\partial_{u_{0,K'}} q_{0,0}^{(1)} = 0$. This shows that the function $q_{0,0}^{(1)}$ cannot depend on $u_{0,j}$ with $j \neq 0$.

The proof for the other cases is quite similar. ∎

As we cannot use the formal variational derivative, we have to work directly with functions $q_{0,0}^{(1)}, q_{0,0}^{(2)}, p_{0,0}^{(3)}, p_{0,0}^{(4)}$ which have the following structure :

$$\begin{aligned}
q_{0,0}^{(m)} &= Q^{(m)}(u_{N_m,0}, \ldots, u_{N'_m,0}), \qquad && m = 1, 2;\\
p_{0,0}^{(l)} &= P^{(l)}(u_{0,K_l}, \ldots, u_{0,K'_l}), && l = 3, 4.
\end{aligned}$$

In Section 5 we are going to limit ourselves to just 5 points symmetries. This will make the problem more definite in the sense that the numbers N_m, N'_m, K_l, K'_l will be specified and small.

5 Integrability conditions for 5 points symmetries

From the definition of Lie symmetry, we can construct a new symmetry by adding the right hand sides of two symmetries $u_{0,0,t} = g_{0,0}$ and $u_{0,0,\tilde{t}} = \tilde{g}_{0,0}$: $u_{0,0,\hat{t}} = \hat{g}_{0,0} = c_1 g_{0,0} + c_2 \tilde{g}_{0,0}$, where c_1, c_2 are arbitrary constants. For example, eq. (26) of Section 3 has two 3 points symmetries (27) and (28), therefore it has a 5 points generalized symmetry:

$$u_{0,0,t} = g_{0,0} = G(u_{1,0}, u_{-1,0}, u_{0,0}, u_{0,1}, u_{0,-1}), \quad g_{u_{1,0}} g_{u_{-1,0}} g_{u_{0,1}} g_{u_{0,-1}} \neq 0. \tag{45}$$

The other known integrable examples of the form (6) have also 5 points generalized symmetries. We are going to use the existence of a 5 points generalized symmetry of the form (45) as an *integrability criteria*. This may be a severe restriction, as there might be integrable equations with symmetries depending on more lattice points.

In the ABS classification all 3 points generalized symmetries turn out to be Miura transformations of the Volterra equation or of the Yamilov discretization of the Krichever–Novikov equation [26]. If we expect to find new type integrable discrete equations of the form (6) these should have as generalized symmetries some new type integrable equations. One example of such equation is given by the Narita-Itoh-Bogoyavlensky [13, 23, 32] equation

$$u_{0,0,t} = g_{0,0} = u_{0,0}(u_{2,0} + u_{1,0} - u_{-1,0} - u_{-2,0}). \tag{46}$$

We will prove in the Appendix that no equation of the form (6) can have eq. (46) as a symmetry.

We can then state the following Theorem:

Theorem 5 *If eq. (6, 8) possesses a generalized symmetry of the form (45), then the functions*

$$\begin{array}{ll} q_{0,0}^{(m)} = Q^{(m)}(u_{2,0}, u_{1,0}, u_{0,0}), & m = 1, 2; \\ p_{0,0}^{(m)} = P^{(m)}(u_{0,2}, u_{0,1}, u_{0,0}), & m = 3, 4, \end{array} \tag{47}$$

satisfy the conditions (39, 40–43).

Proof. From the relations (33–36), as $n = k = 1$ and $n' = k' = -1$, we are able to construct the functions:

$$\begin{array}{ll} q_{0,0}^{(1)} = -T_1 \log g_{u_{1,0}}, & q_{0,0}^{(2)} = T_1 \log g_{u_{-1,0}}, \\ p_{0,0}^{(3)} = -T_2 \log g_{u_{0,1}}, & p_{0,0}^{(4)} = T_2 \log g_{u_{0,-1}}, \end{array} \tag{48}$$

satisfying conditions (39, 40–43). It follows from eqs. (22, 45) that the function $q_{0,0}^{(1)}$ has the structure:

$$q_{0,0}^{(1)} = \hat{Q}^{(1)}(u_{2,0}, u_{1,0}, u_{0,0}, u_{1,1}, u_{1,-1}) = Q^{(1)}(u_{2,0}, u_{1,0}, u_{0,0}, u_{0,1}, u_{0,-1}).$$

It analogy to Theorem 4 we get that $Q^{(1)}$ cannot depend on $u_{0,1}, u_{0,-1}$. The proof for the other functions contained in eqs. (48) is obtained in the same way. ∎

So, for a given eq. (6), we check the integrability conditions (39, 40–43) with the unknown functions $q_{0,0}^{(m)}$ and $p_{0,0}^{(m)}$ given in the form (47). If the integrability conditions are satisfied, we can construct the most general unknown functions $q_{0,0}^{(m)}$ and $p_{0,0}^{(m)}$ of the form (47) and then, from eqs. (48), build the partial derivatives of $g_{0,0}$. The partial derivatives of $g_{0,0}$ must be consistent. The consistency of eqs. (48) imply that the *additional integrability conditions*

$$g_{u_{1,0},u_{-1,0}} = g_{u_{-1,0},u_{1,0}}, \qquad g_{u_{0,1},u_{0,-1}} = g_{u_{0,-1},u_{0,1}} \tag{49}$$

must be satisfied. If eqs. (49) are satisfied, we obtain the right hand side of the symmetry (45) up to an arbitrary unknown function of $u_{0,0}$ of the form $\phi(u_{0,0})$. The function ϕ is derived by using the compatibility condition (12), the *final integrability condition*.

The function $g_{0,0}$, so obtained, will thus be of the form:

$$g_{0,0} = \Phi(u_{1,0}, u_{0,0}, u_{-1,0}) + \Psi(u_{0,1}, u_{0,0}, u_{0,-1}), \tag{50}$$

i.e. the right hand side of any 5 points symmetry (45) must have the form (50). The same result has been obtained by Rasin and Hydon in [35].

All known integrable autonomous equations (6) have symmetries of the following two types:

$$\Psi = 0 \qquad \text{and} \qquad \Phi_{u_{1,0}} \Phi_{u_{-1,0}} \neq 0; \tag{51}$$

$$\Phi = 0 \qquad \text{and} \qquad \Psi_{u_{0,1}} \Psi_{u_{0,-1}} \neq 0. \tag{52}$$

Thus any symmetry of the form (45, 50) is the linear combination of a symmetry (51) and (52). However, we cannot prove this property theoretically.

Obviously, the scheme described in this and in the previous Sections can also be applied to the simpler symmetries (51) and (52). For example, in case of a symmetry given by eqs. (50, 51), just the integrability conditions (39, 40, 41) must be satisfied. The first two equations of eq. (48) allow us to construct the partial derivatives of $g_{0,0} = \Phi$. Then we check the first of the conditions (49). If it is satisfied, we can find Φ up to an arbitrary function $\phi(u_{0,0})$, which can be specified by using the compatibility condition (12).

In the case of the example considered in eq. (26) in Section 3, it is easy to check that the conditions (39, 40–43) are satisfied. Moreover, using the generalized symmetries (27, 28) and eqs. (48), we easily construct four conservation laws which are linear combinations with shift dependent parameters of the conservation laws (29, 30).

It is worthwhile to notice that integrability conditions analogous to eqs. (39, 40–43) have been derived for hyperbolic systems of the form (4) by Zhiber and Shabat in [48].

6 A simple classification problem

Here we apply the formulae introduced before to study the class of equations:

$$u_{1,1} = f_{0,0} = u_{1,0} + u_{0,1} + \varphi(u_{0,0}). \tag{53}$$

The class of equations (53) depends on an unknown function φ, and we require that eq. (53) possess a generalized symmetry of the form (45). To do so it must satisfy the integrability conditions (39, 40–43, 47). If $\varphi'' = 0$, equation (53) is linear, and all the integrability conditions are satisfied trivially. So we require that $\varphi'' \neq 0$.

The proof that eqs. (39, 40–43, 47) are conservation laws is carried out by differentiating them in such a way to reduce them to simple differential equations, a scheme introduced in 1823 by Abel [1] (see [3] for a review) for solving functional equations. The applications of this scheme for difference equation can be found in [21, 27, 36]. In [36] the scheme was used for finding conservation laws for known equations, i.e. when the dependence of the functions $p_{0,0}$ and $q_{0,0}$ on the symmetries and on the equation (6) was unknown while the difference equation (6) was given. In [37] the existence of a simple conservation law is used as an integrability condition.

Here we consider the case when either $p_{0,0}$ or $q_{0,0}$ is expressed in terms of the unknown right hand side of eq. (6). The conservation laws are allowed to depend on arbitrary functions of the variables $u_{1,0}, u_{0,0}, u_{0,1}$. Moreover, as it will be shown at the end of this Section, the existence of simple conservation laws is not sufficient to prove integrability. One can have nonlinear equations of this class (53) with two local conservation laws but with no generalized symmetry.

Let us study the class of difference equations (53). For later use we can rewrite eq. (53) in three equivalent forms, applying to it the operators T_1^{-1}, T_2^{-1}:

$$\begin{array}{rcl} u_{-1,1} & = & u_{0,1} - u_{0,0} - \varphi(u_{-1,0}), \\ u_{1,-1} & = & u_{1,0} - u_{0,0} - \varphi(u_{0,-1}), \\ u_{-1,-1} & = & \varphi^{-1}(u_{0,0} - u_{-1,0} - u_{0,-1}). \end{array} \tag{54}$$

Let us consider condition (39) with $m = 2$. Applying the shift operators T_1^{-1}, T_2^{-1}, we rewrite it in two equivalent forms:

$$p^{(2)}_{0,0} - p^{(2)}_{-1,0} = q^{(2)}_{-1,1} - q^{(2)}_{-1,0}, \tag{55}$$

$$p^{(2)}_{0,-1} - p^{(2)}_{-1,-1} = q^{(2)}_{-1,0} - q^{(2)}_{-1,-1}, \tag{56}$$

where $p^{(2)}_{0,0} = \log \varphi'(u_{0,0})$ and $q^{(2)}_{0,0}$ is given by eq. (47). Taking into account eqs. (53, 54), eqs. (55, 56) can be expressed in terms of the independent variables (9).

Eqs. (55, 56) are two functional equations for $q^{(2)}_{0,0}$. By applying the following operators:

$$\hat{\mathcal{A}} = \partial_{u_{0,0}} + \partial_{u_{1,0}} + \partial_{u_{-1,0}}, \qquad \hat{\mathcal{B}} = \partial_{u_{0,0}} - \varphi'(u_{0,0})\partial_{u_{1,0}} - \frac{1}{\varphi'(u_{-1,0})}\partial_{u_{-1,0}},$$

we reduce them to partial differential equations. Using eqs. (54), we can show that $\hat{\mathcal{A}}$ annihilates any function $\Phi(u_{1,-1}, u_{0,-1}, u_{-1,-1})$. So, applying $\hat{\mathcal{A}}$ to eq. (56), we get:

$$\hat{\mathcal{A}}\ q^{(2)}_{-1,0} = 0. \tag{57}$$

The operator $\hat{\mathcal{B}}$ annihilates $q^{(2)}_{-1,1}$. Thus applying the operator $\hat{\mathcal{B}}$ to eq. (55) we get:

$$\hat{\mathcal{B}}\ q^{(2)}_{-1,0} = -\hat{\mathcal{B}}\ (p^{(2)}_{0,0} - p^{(2)}_{-1,0}).$$

If we introduce the difference operator $\hat{\mathcal{C}} = \hat{\mathcal{A}} - \hat{\mathcal{B}}$, we get

$$\hat{\mathcal{C}}\ q^{(2)}_{-1,0} = \hat{\mathcal{B}}\ (p^{(2)}_{0,0} - p^{(2)}_{-1,0}). \tag{58}$$

From eqs. (57, 58) we also get:

$$[\hat{\mathcal{A}}, \hat{\mathcal{C}}]\ q^{(2)}_{-1,0} = \hat{\mathcal{A}}\hat{\mathcal{B}}\ (p^{(2)}_{0,0} - p^{(2)}_{-1,0}), \tag{59}$$

where $[\hat{\mathcal{A}}, \hat{\mathcal{C}}]$ is the standard commutator of two operators. So eqs. (57–59) can be rewritten as a partial differential system for the function $q = q^{(2)}_{-1,0}$, where, as before, by the indexes we denote partial derivatives and by apices derivatives with respect to the argument:

$$\begin{aligned} q_{u_{0,0}} + q_{u_{1,0}} + q_{u_{-1,0}} &= 0, \\ a(u_{0,0})q_{u_{1,0}} + b(u_{-1,0})q_{u_{-1,0}} &= c(u_{0,0}) - b'(u_{-1,0}), \\ a'(u_{0,0})q_{u_{1,0}} + b'(u_{-1,0})q_{u_{-1,0}} &= c'(u_{0,0}) - b''(u_{-1,0}). \end{aligned} \tag{60}$$

The functions $a(z)$, $b(z)$ and $c(z)$ are given by

$$a(z) = \varphi'(z) + 1, \quad b(z) = \frac{1}{\varphi'(z)} + 1, \quad c(z) = \frac{\varphi''(z)}{\varphi'(z)},$$

where $a'(z)b'(z)c(z) \neq 0$, as $\varphi''(z) \neq 0$.

The solvability of the system (60) depends on the following determinant:

$$\Delta = \begin{vmatrix} a(u_{0,0}) & b(u_{-1,0}) \\ a'(u_{0,0}) & b'(u_{-1,0}) \end{vmatrix}.$$

We must have $\Delta \neq 0$. If we have $\Delta = 0$, as $u_{0,0}$ and $u_{-1,0}$ are independent variables, we obtain the relations $\frac{a'(u_{0,0})}{a(u_{0,0})} = \frac{b'(u_{-1,0})}{b(u_{-1,0})} = \nu$, where ν is a constant. These relations are in contradiction with the condition that $\varphi'' \neq 0$.

If we differentiate the system (60) with respect to $u_{1,0}$, we easily deduce that $q_{u_{1,0}} = \alpha$, where α is a constant. Then from eqs. (60) we obtain two different expressions for $q_{u_{-1,0}}$:

$$q_{u_{-1,0}} = \frac{d(u_{0,0}) - b'(u_{-1,0})}{b(u_{-1,0})} = \frac{d'(u_{0,0}) - b''(u_{-1,0})}{b'(u_{-1,0})}, \qquad d(z) = c(z) - \alpha a(z). \tag{61}$$

If $d' \neq 0$, differentiating eq. (61) with respect to $u_{0,0}$, we get $\frac{d''(u_{0,0})}{d'(u_{0,0})} = \frac{b'(u_{-1,0})}{b(u_{-1,0})} = \sigma$, where σ is a constant. This result is again in contradiction with the condition $\varphi'' \neq 0$. So, $d = \beta$, a constant, and we get the following ODE for φ:

$$\varphi''/\varphi' = \alpha\varphi' + \alpha + \beta. \tag{62}$$

If φ satisfies eq. (62), the condition (61) is satisfied, and $q_{u_{-1,0}} = \alpha + \beta$.

The system (60) provides us with another partial derivative of q:

$$q_{u_{0,0}} = -2\alpha - \beta,$$

from which we deduce that

$$q = q^{(2)}_{-1,0} = \alpha u_{1,0} - (2\alpha + \beta)u_{0,0} + (\alpha + \beta)u_{-1,0} + \delta,$$

where δ is an arbitrary constant. The integration of eq. (62) gives:

$$\log \varphi'(z) = \alpha\varphi(z) + (\alpha + \beta)z + \gamma,$$

where γ is a further constant. If we introduce these last two equations into eq. (55), we get $\beta u_{0,0} + \beta\varphi(u_{-1,0}) = 0$, which implies $\beta = 0$.

Thus, we have proved that eq. (53) satisfies the condition (39) with $m = 2$ if and only if

$$\log \varphi'(z) = \alpha(\varphi(z) + z) + \gamma, \tag{63}$$

with $\alpha \neq 0$, as $\varphi'' \neq 0$. If equation (63) is satisfied,

$$p^{(2)}_{0,0} = \log \varphi'(u_{0,0}), \qquad q^{(2)}_{0,0} = \alpha(u_{2,0} - 2u_{1,0} + u_{0,0}) + \delta, \tag{64}$$

and these functions define a nontrivial conservation law.

If eq. (53), with φ given by eq. (63), has a generalized symmetry of the form (45), the other conditions (39, 40–43, 47) must be satisfied. From eq. (40) we get that the condition (39) with $m = 1$ becomes $(T_2 - 1)q^{(1)}_{0,0} = 0$. This equation has a trivial solution, $q^{(1)}_{0,0}$ a constant. We now look for a nontrivial solution. From eqs. (47) it follows that the functions $q^{(1)}_{0,0}$ and $q^{(2)}_{0,0}$ depend on the same set of variables. Hence $\tilde{q} = q^{(1)}_{-1,0}$ also satisfies eqs. (60), but with zeros on the right hand side. As $q_{u_{1,0}}$ is a constant, it follows that also $q^{(1)}_{0,0}$ must be a constant, i.e. the constant solution is the most general one. From eqs. (48), we get the partial derivatives of the right hand side of the symmetry (45), $g_{u_{1,0}}$ and $g_{u_{-1,0}}$. It is easy to verify that the first of the conditions (49) is not satisfied. Consequently eq. (53), with φ given by eq. (63), has no generalized symmetry of the form (45).

In Section 5 we have considered the simpler symmetries (51, 52). Using the previous reasoning, we can prove that there is no symmetry defined by eqs. (50, 51). Eq. (53) is symmetric under the involution $u_{i,j} \to u_{j,i}$. Also the conditions (39) with $m = 3, 4$ are symmetric with respect to the conditions (39) with $m = 1, 2$.

So these further conditions will provide a conservation law symmetric to the one defined by eqs. (64) and prove that there is no symmetry given by eqs. (50, 52).

Let us collect the results obtained so far in the following Theorem, where the conservation laws will be written in a simplified form, omitting inessential constants.

Theorem 6 *Eq. (53) satisfies the integrability conditions (39, 40–43, 47) iff φ is a solution of eq. (63). Eq. (53), when φ is given by eq. (63), has two nontrivial conservation laws:*

$$\begin{array}{rcl} (T_1 - 1)(\varphi(u_{0,0}) + u_{0,0}) & = & (T_2 - 1)(u_{2,0} - 2u_{1,0} + u_{0,0}), \\ (T_2 - 1)(\varphi(u_{0,0}) + u_{0,0}) & = & (T_1 - 1)(u_{0,2} - 2u_{0,1} + u_{0,0}). \end{array} \tag{65}$$

However, in this case, eq. (53) does not have a generalized symmetry of the form (45) or of the form given by eqs. (50, 51) or (50, 52).

Let us notice that eq. (53) possesses the conservation laws (65) for any φ, not only when φ satisfies eq. (63). However, the integrability conditions are satisfied only if φ satisfies eq. (63), but no generalized symmetry of the form mentioned in Theorem 6 exists.

7 Conclusions

In this paper we have considered the classification problem for difference equations by asking for the existence of a generalized symmetry. In this way we have obtained the lowest order integrability conditions which turn out to be written as conservation laws. The verification of the existence of finite order conservation laws is in general a very complicated problem due to the high number of unknown involved. So we reduce ourselves to the case when we have just a 5 points symmetry. In this case we easily can find some further integrability conditions which make our problem solvable. At the end we present an example of classification when we have just an arbitrary function of one variable.

This research is far from complete. At the moment we are working on:

1. Obtaining further integrability conditions by adding extra structures;
2. Applying the result contained in this work for testing the integrability of some discrete equations of the class (1) as, for example, Q_V [42];
3. Classifying eqs. (1) in the case of an arbitrary function of two variables.

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APPENDIX

Theorem 7 *No equation of the form (6, 8) can have a generalized symmetry of the form of eq. (46).*

Proof. We use conditions (33, 34) with $n = 2$ and $n' = -2$. Applying the operators T_1^{-1} and $-T_1$, we rewrite them in the form:

$$p_{1,0}^{(1)} - p_{-1,0}^{(1)} = \log \frac{u_{0,0}}{u_{0,1}}, \tag{66}$$

$$p_{1,0}^{(2)} - p_{-1,0}^{(2)} = \log \frac{u_{1,1}}{u_{1,0}}, \tag{67}$$

where $p_{0,0}^{(1)}$, $p_{0,0}^{(2)}$ are given by eqs. (40, 41). Studying conditions (66, 67), we will use in addition to eq. (6) its equivalent form:

$$u_{-1,1} = \hat{f}_{0,0} = \hat{F}(u_{-1,0}, u_{0,0}, u_{0,1}).$$

The functions $p_{0,0}^{(m)}$ have the structure $p_{0,0}^{(m)} = P^{(m)}(u_{1,0}, u_{0,0}, u_{0,1})$. Therefore $p_{-1,0}^{(m)} = P^{(m)}(u_{0,0}, u_{-1,0}, \hat{f}_{0,0})$ and the right hand sides of eqs. (66, 67) do not depend on $u_{2,0}$. The functions $p_{1,0}^{(m)} = P^{(m)}(u_{2,0}, u_{1,0}, f_{0,0})$ depend on $u_{2,0}$, and from eqs. (66, 67) we get: $\partial_{u_{2,0}} p_{1,0}^{(m)} = T_1 \partial_{u_{1,0}} p_{0,0}^{(m)} = 0$. Moreover, according to Proposition 1,

$$\partial_{u_{1,0}} p_{0,0}^{(m)} = 0, \qquad m = 1, 2. \tag{68}$$

From eq. (68) with $m = 1$ we get $f_{u_{1,0}u_{1,0}} = 0$, i.e. $f_{0,0}$ can be expressed as:

$$f_{0,0} = a_{0,0} u_{1,0} + b_{0,0} = A(u_{0,0}, u_{0,1}) u_{1,0} + B(u_{0,0}, u_{0,1}), \tag{69}$$

where $a_{0,0} \neq 0$ due to condition (8). Now $p_{0,0}^{(1)} = \log a_{0,0}$ and eq. (66) is rewritten as:

$$\frac{a_{1,0}}{a_{-1,0}} = \frac{u_{0,0}}{u_{0,1}}. \tag{70}$$

Here only the function $a_{1,0}$ depends on $u_{1,0}$, and we get:

$$\frac{da_{1,0}}{du_{1,0}} = \partial_{u_{1,0}} a_{1,0} + a_{0,0} \partial_{u_{1,1}} a_{1,0} = 0.$$

Applying to it the shift operator T_1^{-1}, we get the more convenient form:

$$\partial_{u_{0,0}} a_{0,0} + a_{-1,0} \partial_{u_{0,1}} a_{0,0} = 0. \tag{71}$$

As $a_{-1,0} \neq 0$, only two cases are possible. The first one is when $\partial_{u_{0,0}} a_{0,0} = \partial_{u_{0,1}} a_{0,0} = 0$, i.e. $a_{0,0}$ is a constant. This is in contradiction with eq. (70). So, $\partial_{u_{0,0}} a_{0,0} \neq 0$ and $\partial_{u_{0,1}} a_{0,0} \neq 0$.

From eq. (68) with $m=2$ we get

$$\frac{f_{u_{0,0}u_{1,0}}}{f_{u_{0,0}}} - \frac{f_{u_{0,1}u_{1,0}}}{f_{u_{0,1}}} = 0.$$

Using this equation together with eqs. (69, 71), we get:

$$p_{0,0}^{(2)} = \log \frac{f_{u_{0,0}u_{1,0}}}{f_{u_{0,1}u_{1,0}}} = \log \frac{\partial_{u_{0,0}} a_{0,0}}{\partial_{u_{0,1}} a_{0,0}} = \log(-a_{-1,0}).$$

Applying T_1 we can rewrite eq. (67) as:

$$\frac{a_{1,0}}{a_{-1,0}} = \frac{u_{2,1}}{u_{2,0}}. \tag{72}$$

Comparing eqs. (70, 72) and using eq. (69), we get $u_{2,1} = \frac{u_{0,0}}{u_{0,1}} u_{2,0} = a_{1,0}u_{2,0} + b_{1,0}$. As $a_{1,0}$, $b_{1,0}$ do not depend on $u_{2,0}$, we obtain from here $a_{1,0} = \frac{u_{0,0}}{u_{0,1}}$. Then from eq. (70) we obtain $a_{-1,0} = 1$. These two last results are in contradiction, thus proving the Theorem. ∎

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